

Sherali-Adams relaxations for valued CSPs*

Johan Thapper
 Université Paris-Est, Marne-la-Vallée, France
 thapper@u-pem.fr

Stanislav Živný
 University of Oxford, UK
 standa@cs.ox.ac.uk

Abstract

We consider Sherali-Adams linear programming relaxations for solving valued constraint satisfaction problems to optimality. The utility of linear programming relaxations in this context have previously been demonstrated using the lowest possible level of this hierarchy under the name of the basic linear programming relaxation (BLP). It has been shown that valued constraint languages containing only finite-valued weighted relations are tractable if, and only if, the integrality gap of the BLP is 1. In this paper, we demonstrate that almost all of the known tractable languages with arbitrary weighted relations have an integrality gap 1 for the Sherali-Adams relaxation with parameters $(2, 3)$. The result is closely connected to the notion of bounded relational width for the ordinary constraint satisfaction problem and its recent characterisation.

1 Introduction

The constraint satisfaction problem provides a common framework for many theoretical and practical problems in computer science. An instance of the *constraint satisfaction problem* (CSP) consists of a collection of variables that must be assigned labels from a given domain subject to specified constraints. The CSP is NP-complete in general, but tractable fragments can be studied by, following Feder and Vardi [13], restricting the constraint relations allowed in the instances to a fixed, finite set, called the constraint language. The most successful approach to classifying the language-restricted CSP is the so-called algebraic approach [3, 5].

An important type of algorithms for CSPs are *consistency methods*. A constraint language is of *bounded relational width* if any CSP instance over this language can be solved by establishing (k, ℓ) -minimality for some fixed integers $1 \leq k \leq \ell$ [1]. The power of consistency methods for constraint languages has recently been fully characterised [3, 21] and it has been shown that any constraint language that is of bounded relational width is of relational width at most $(2, 3)$ [1].

The CSP deals with only feasibility issues: Is there a solution satisfying certain constraints? In this work we are interested in problems that capture both feasibility and optimisation issues: What is the best solution satisfying certain constraints? Problems of this form can be cast as valued constraint satisfaction problems [16].

An instance of the *valued constraint satisfaction problem* (VCSP) is given by a collection of variables that is assigned labels from a given domain with the goal to *minimise* an objective function given by a sum of weighted relations, each depending on some subset of the variables [8]. The weighted relations can take on finite rational values and positive infinity. The CSP corresponds to the special case of the VCSP when the codomain of all weighted relations is $\{0, \infty\}$.

*The authors were supported by a London Mathematical Society Grant. Stanislav Živný was supported by a Royal Society University Research Fellowship.

Like the CSP, the VCSP is NP-hard in general and thus we are interested in the restrictions which give rise to tractable classes of problems. We restrict the *valued constraint language*; that is, all weighted relations in a given instance must belong to a fixed set of weighted relations on the domain. The ultimate goal is to understand the computational complexity of all valued constraint languages, that is, determine which languages give rise to classes of problems solvable in polynomial time and which languages give rise to classes of problems that are NP-hard. Languages of the former type are called *tractable*, and languages of the latter type are called *intractable*. The computational complexity of Boolean (on a 2-element domain) valued constraint languages [8] and conservative (containing all $\{0, 1\}$ -valued unary weighted relations) valued constraint languages [18] have been completely classified with respect to exact solvability.

Every VCSP problem has a natural linear programming (LP) relaxation, proposed independently by a number of authors, e.g. [6], and referred to as the *basic* LP relaxation (BLP) of the VCSP. It is the first level in the Sherali-Adams hierarchy [24], which provides successively tighter LP relaxations of an integer LP. The BLP has been considered in the context of CSPs for robust approximability [10, 20] and constant-factor approximation [9, 12]. Higher levels of Sherali-Adams hierarchy have been considered for (in)approximability of CSPs [11, 31] but we are not aware of any results related to exact solvability of (valued) CSPs. Semidefinite programming relaxations have also been considered in the context of CSPs for approximability [23] and robust approximability [2].

Consistency methods, and in particular strong 3-consistency has played an important role as a preprocessing step in establishing tractability of valued constraint languages. Cohen et al. proved the tractability of valued constraint languages improved by a symmetric tournament pair (STP) multimorphism via strong 3-consistency preprocessing, and an involved reduction to submodular function minimisation [7]. They also showed that the tractability of any valued constraint language improved by a tournament pair multimorphism via a preprocessing using results on constraint languages invariant under a 2-semilattice polymorphism, which relies on $(3, 3)$ -minimality, and then reducing to the STP case. The only tractable conservative valued constraint languages are those admitting a pair of fractional polymorphisms called STP and MJN [18]; again, the tractability of such languages is proved via a 3-consistency preprocessing reducing to the STP case. It is natural to ask whether this nested use of consistency methods are necessary.

1.1 Contributions

In [17, 26], the authors showed that the BLP of the VCSP can be used to solve the problem for many valued constraint languages. In [27], it was then shown that for VCSPs with weighted relations taking only finite values, the BLP precisely characterises the tractable (finite-)valued constraint languages; i.e., if BLP fails to solve any instance of some valued constraint language of this type, then this language is NP-hard.

In this paper, we show that a higher-level Sherali-Adams linear programming relaxation [24] suffices to solve most of the previously known tractable valued constraint languages with arbitrary weighted relations, and in particular, all known valued constraint languages that involve some optimisation (and thus do not reduce to constraint languages containing only relations) except for valued constraint languages of generalised weak tournament pair type [30]; such languages are known to be tractable [30] but we do not know whether they are tractable by our linear programming relaxation.

Our main result, Theorem 4, shows that if the support clone of a valued constraint language Γ of finite size contains weak near-unanimity operations of all but finitely many arities, then Γ is tractable via the Sherali-Adams relaxation with parameters $(2, 3)$. This tractability condition is precisely the bounded relational width condition for constraint languages of finite size containing

all constants [3, 21], and our proof fundamentally relies on the results of Barto and Kozik [3] and Barto [1].

It is folklore that the k th level of Sherali-Adams hierarchy establishes k -consistency for CSPs. We demonstrate that one linear programming relaxation is powerful enough to establish consistency as well as solving an optimisation problem in one go without the need of nested applications of consistency methods. For example, valued constraint languages having a tournament pair polymorphism were previously known to be tractable using ingenious application of various consistency techniques, advanced analysis of constraint networks using modular decompositions, and submodular function minimisation [7]. Here, we show that an even less restrictive condition (having a binary conservative commutative operation in some fractional polymorphism) ensures that the Sherali-Adams relaxation solves all instances to optimum.

Finally, we also give a short proof of the dichotomy theorem for conservative valued constraint languages [18], which previously needed lengthy arguments (although we still rely on Takhanov [25] for a part of the proof).

2 Preliminaries

2.1 Valued CSPs

Throughout the paper, let D be a fixed finite set of size at least two.

Definition 1. *An m -ary relation over D is any mapping $\phi : D^m \rightarrow \{c, \infty\}$ for some $c \in \mathbb{Q}$. We denote by \mathbf{R}_D the set of all relations on D .¹*

Let $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ denote the set of rational numbers with (positive) infinity.

Definition 2. *An m -ary weighted relation over D is any mapping $\phi : D^m \rightarrow \overline{\mathbb{Q}}$. We write $\text{ar}(\phi) = m$ for the arity of ϕ . We denote by Φ_D the set of all weighted relations on D .*

For any m -ary weighted relation $\phi \in \Phi_D$, we denote by $\text{Feas}(\phi) = \{\mathbf{x} \in D^m \mid \phi(\mathbf{x}) < \infty\} \in \mathbf{R}_D$ the underlying m -ary *feasibility relation*, and by $\text{Opt}(\phi) = \{\mathbf{x} \in \text{Feas}(\phi) \mid \forall \mathbf{y} \in D^m : \phi(\mathbf{x}) \leq \phi(\mathbf{y})\} \in \mathbf{R}_D$ the m -ary *optimality relation*, which contains the tuples on which ϕ is minimised. A weighted relation $\phi : D^m \rightarrow \overline{\mathbb{Q}}$ is called *finite-valued* if $\text{Feas}(\phi) = D^m$.

Definition 3. *Let $V = \{x_1, \dots, x_n\}$ be a set of variables. A valued constraint over V is an expression of the form $\phi(\mathbf{x})$ where $\phi \in \Phi_D$ and $\mathbf{x} \in V^{\text{ar}(\phi)}$. The number m is called the *arity* of the constraint, the weighted relation ϕ is called the *constraint weighted relation*, and the tuple \mathbf{x} the *scope* of the constraint.*

We call D the *domain*, the elements of D *labels* and say that weighted relations take *values*.

Definition 4. *An instance of the valued constraint satisfaction problem (VCSP) is specified by a finite set $V = \{x_1, \dots, x_n\}$ of variables, a finite set D of labels, and an objective function I expressed as follows:*

$$I(x_1, \dots, x_n) = \sum_{i=1}^q \phi_i(\mathbf{x}_i), \quad (1)$$

¹ An m -ary relation over D is commonly defined as a subset of D^m . Note that Definition 1 is equivalent to the standard definition as any mapping $\phi : D^m \rightarrow \{c, \infty\}$ represents the set $R = \{\mathbf{x} \in D^m \mid \phi(\mathbf{x}) < \infty\}$ and any set $R \subseteq D^m$ can be represented by $\phi_R : D^m \rightarrow \{0, \infty\}$ defined by $\phi_R(\mathbf{x}) = 0$ for $\mathbf{x} \in R$ and $\phi_R(\mathbf{x}) = \infty$ for $\mathbf{x} \notin R$. Consequently, we shall use both definitions interchangeably.

where each $\phi_i(\mathbf{x}_i)$, $1 \leq i \leq q$, is a valued constraint over V . Each constraint can appear multiple times in I . The goal is to find an assignment (or solution) of labels to the variables minimising I .

A solution is called *feasible* (or *satisfying*) if it is of finite value. A VCSP instance I is called *satisfiable* if there is a feasible solution to I . CSPs are a special case of VCSPs with (unweighted) relations with the goal to determine the existence of a feasible solution.

Example 1. In the MIN-UNCUT problem the goal is to find a partition of the vertices of a given graph into two parts so that the number of edges inside the two partitions is minimised. For a graph (V, E) with $V = \{x_1, \dots, x_n\}$, this NP-hard problem can be expressed as the VCSP instance $I(x_1, \dots, x_n) = \sum_{(i,j) \in E} \phi_{\text{xor}}(x_i, x_j)$ over the Boolean domain $D = \{0, 1\}$, where $\phi_{\text{xor}} : \{0, 1\}^2 \rightarrow \overline{\mathbb{Q}}$ is defined by $\phi_{\text{xor}}(x, y) = 1$ if $x = y$ and $\phi_{\text{xor}}(x, y) = 0$ if $x \neq y$.

Definition 5. Any set $\Delta \subseteq \mathbf{R}_D$ is called a constraint language over D . Any set $\Gamma \subseteq \Phi_D$ is called a valued constraint language over D . We denote by $\text{VCSP}(\Gamma)$ the class of all VCSP instances in which the constraint weighted relations are all contained in Γ .

For a constraint language Δ , we denote by $\text{CSP}(\Delta)$ the class $\text{VCSP}(\Delta)$ to emphasise the fact that there is no optimisation involved.

Definition 6. A valued constraint language Γ is called *tractable* if $\text{VCSP}(\Gamma')$ can be solved (to optimality) in polynomial time for every finite subset $\Gamma' \subseteq \Gamma$, and Γ is called *intractable* if $\text{VCSP}(\Gamma')$ is NP-hard for some finite $\Gamma' \subseteq \Gamma$.

Example 1 shows that the valued constraint language $\{\phi_{\text{xor}}\}$ is intractable.

2.2 Operations and Clones

We recall some basic terminology from universal algebra. Given an m -tuple $\mathbf{x} \in D^m$, we denote its i th entry by $\mathbf{x}[i]$ for $1 \leq i \leq m$. Any mapping $f : D^k \rightarrow D$ is called a k -ary operation; f is called *conservative* if $f(x_1, \dots, x_k) \in \{x_1, \dots, x_k\}$ and *idempotent* if $f(x, \dots, x) = x$. We will apply a k -ary operation f to k m -tuples $\mathbf{x}_1, \dots, \mathbf{x}_k \in D^m$ coordinatewise, that is,

$$f(\mathbf{x}_1, \dots, \mathbf{x}_k) = (f(\mathbf{x}_1[1], \dots, \mathbf{x}_k[1]), \dots, f(\mathbf{x}_1[m], \dots, \mathbf{x}_k[m])). \quad (2)$$

Definition 7. Let ϕ be an m -ary weighted relation on D . A k -ary operation f on D is a polymorphism of ϕ if, for any $\mathbf{x}_1, \dots, \mathbf{x}_k \in D^m$ with $\mathbf{x}_i \in \text{Feas}(\phi)$ for all $1 \leq i \leq k$, we have that $f(\mathbf{x}_1, \dots, \mathbf{x}_k) \in \text{Feas}(\phi)$.

For any valued constraint language Γ over a set D , we denote by $\text{Pol}(\Gamma)$ the set of all operations on D which are polymorphisms of all $\phi \in \Gamma$. We write $\text{Pol}(\phi)$ for $\text{Pol}(\{\phi\})$.

A k -ary projection is an operation of the form $\pi_i^{(k)}(x_1, \dots, x_k) = x_i$ for some $1 \leq i \leq k$. Projections are polymorphisms of all valued constraint languages.

The *composition* of a k -ary operation $f : D^k \rightarrow D$ with k ℓ -ary operations $g_i : D^\ell \rightarrow D$ for $1 \leq i \leq k$ is the ℓ -ary function $f[g_1, \dots, g_k] : D^\ell \rightarrow D$ defined by

$$f[g_1, \dots, g_k](x_1, \dots, x_\ell) = f(g_1(x_1, \dots, x_\ell), \dots, g_k(x_1, \dots, x_\ell)). \quad (3)$$

We denote by \mathcal{O}_D the set of all finitary operations on D and by $\mathcal{O}_D^{(k)}$ the k -ary operations in \mathcal{O}_D .

A *clone* of operations, $C \subseteq \mathcal{O}_D$, is a set of operations on D that contains all projections and is closed under composition. It is easy to show that $\text{Pol}(\Gamma)$ is a clone for any valued constraint language Γ .

Definition 8. A k -ary fractional operation ω is a probability distribution over $\mathcal{O}_D^{(k)}$. We define $\text{supp}(\omega) = \{f \in \mathcal{O}_D^{(k)} \mid \omega(f) > 0\}$.

Definition 9. Let ϕ be an m -ary weighted relation on D and let ω be a k -ary fractional operation on D . We call ω a fractional polymorphism of ϕ if $\text{supp}(\omega) \subseteq \text{Pol}(\phi)$ and for any $\mathbf{x}_1, \dots, \mathbf{x}_k \in D^m$ with $\mathbf{x}_i \in \text{Feas}(\phi)$ for all $1 \leq i \leq k$, we have

$$\mathbb{E}_{f \sim \omega} [\phi(f(\mathbf{x}_1, \dots, \mathbf{x}_k))] \leq \text{avg}\{\phi(\mathbf{x}_1), \dots, \phi(\mathbf{x}_k)\}. \quad (4)$$

We also say that ϕ is improved by ω .

Definition 10. For any valued constraint language $\Gamma \subseteq \Phi_D$, we define $\text{fPol}(\Gamma)$ to be the set of all fractional operations that are fractional polymorphisms of all weighted relations $\phi \in \Gamma$. We write $\text{fPol}(\phi)$ for $\text{fPol}(\{\phi\})$.

Example 2. A valued constraint language on domain $\{0, 1\}$ is called *submodular* if it has the fractional polymorphism ω defined by $\omega(\min) = \omega(\max) = \frac{1}{2}$, where \min and \max are the two binary operations that return the smaller and larger of its two arguments respectively with respect to the usual order $0 < 1$.

Definition 11. Let Γ be a valued constraint language on D . We define

$$\text{supp}(\Gamma) = \bigcup_{\omega \in \text{fPol}(\Gamma)} \text{supp}(\omega). \quad (5)$$

Lemma 1. Let Γ be a valued constraint language of finite size. Then, $\text{supp}(\Gamma)$ is a clone.

We note that Lemma 1 has also been observed in [22] and in [14].

Proof. Observe that $\text{supp}(\Gamma)$ contains all projections as $\tau_k \in \text{fPol}(\Gamma)$ for every $k \geq 1$, where τ_k is the fractional operation defined by $\tau_k(\pi_i^{(k)}) = \frac{1}{k}$ for every $1 \leq i \leq k$. Thus we only need to show that $\text{supp}(\Gamma)$ is closed under composition.

Since $\omega \in \text{supp}(\Gamma)$ there is k -ary $\omega \in \text{fPol}(\Gamma)$ with $\omega(f) > 0$. Moreover, since $g_1, \dots, g_k \in \text{supp}(\Gamma)$, for every $1 \leq i \leq k$ there is ℓ -ary $\mu_i \in \text{supp}(\Gamma)$ with $\mu_i(g_i) > 0$. We define an ℓ -ary fractional operation

$$\omega'(p) = \Pr_{\substack{t \sim \omega \\ h_i \sim \mu_i}} [t[h_1, \dots, h_k] = p]. \quad (6)$$

Since $\omega(f) > 0$ and $\mu_i(g_i) > 0$ for all $1 \leq i \leq k$, we have $\omega'(f[g_1, \dots, g_k]) > 0$. A straightforward verification shows that $\omega' \in \text{fPol}(\Gamma)$. Consequently, $f[g_1, \dots, g_k] \in \text{supp}(\Gamma)$. \square

The following lemma is a generalisation of [28, Lemma 5] from arity one to arbitrary arity and from finite-valued to valued constraint languages, but the proof is analogous. A special case has also been observed, in the context of Min-Sol problems [30], by Hannes Uppman.²

Lemma 2. Let Γ be a valued constraint language of finite size on a domain D and let $f \in \text{Pol}(\Gamma)$. Then, $f \in \text{supp}(\Gamma)$ if, and only if, $f \in \text{Pol}(\text{Opt}(I))$ for all instances I of $\text{VCSP}(\Gamma)$.

²Private communication.

Proof. The operation f is in $\text{supp}(\Gamma)$ if, and only if, there exists a fractional polymorphism ω with $f \in \text{supp}(\omega)$. This is the case if, and only if, the following system of linear inequalities in the variables $\omega(g)$ for $g \in \text{Pol}(\Gamma)$ is satisfiable:

$$\begin{aligned} \sum_{g \in \text{Pol}(\Gamma)} \omega(g) \phi(f(\mathbf{x}_1, \dots, \mathbf{x}_k)) &\leq \text{avg}\{\phi(\mathbf{x}_1), \dots, \phi(\mathbf{x}_k)\}, \quad \forall \phi \in \Gamma, \mathbf{x}_i \in \text{Feas}(\phi), \\ \sum_{g \in \text{Pol}(\Gamma)} \omega(g) &= 1, \\ \omega(f) &> 0, \\ \omega(g) &\geq 0, \quad \forall g \in \text{Pol}(\Gamma). \end{aligned} \tag{7}$$

By Farkas' lemma, the system (7) is unsatisfiable if, and only if, the following system in variables $z(\phi, \mathbf{x}_1, \dots, \mathbf{x}_k)$, for $\phi \in \Gamma, \mathbf{x}_i \in \text{Feas}(\phi)$, is satisfiable:

$$\begin{aligned} \sum_{\phi \in \Gamma, \mathbf{x}_i \in \text{Feas}(\phi)} z(\phi, \mathbf{x}_1, \dots, \mathbf{x}_k) (\text{avg}\{\phi(\mathbf{x}_1), \dots, \phi(\mathbf{x}_k)\} - \phi(g(\mathbf{x}_1, \dots, \mathbf{x}_k))) &\leq 0, \quad \forall g \in \text{Pol}(\Gamma), \\ \sum_{\phi \in \Gamma, \mathbf{x}_i \in \text{Feas}(\phi)} z(\phi, \mathbf{x}_1, \dots, \mathbf{x}_k) (\text{avg}\{\phi(\mathbf{x}_1), \dots, \phi(\mathbf{x}_k)\} - \phi(f(\mathbf{x}_1, \dots, \mathbf{x}_k))) &< 0, \\ z(\phi, \mathbf{x}_1, \dots, \mathbf{x}_k) &\geq 0, \quad \forall \phi \in \Gamma, \mathbf{x}_i \in \text{Feas}(\phi). \end{aligned} \tag{8}$$

First, assume that $f \notin \text{supp}(\Gamma)$ so that (8) has a feasible solution z . Note that by scaling we may assume that z is integral. Then, z can then be interpreted as an instance I_f of $\text{VCSP}(\Gamma)$ in which we take as variables the k -tuples of D , $V = D^k$, and let

$$I_f(\mathbf{x}) = \sum_{\phi \in \Gamma, \mathbf{x}_i \in \text{Feas}(\phi)} z(\phi, \mathbf{x}_1, \dots, \mathbf{x}_k) \phi((\mathbf{x}_1[1], \dots, \mathbf{x}_k[1]), \dots, (\mathbf{x}_1[\text{ar}(\phi)], \dots, \mathbf{x}_k[\text{ar}(\phi)])),$$

where \mathbf{x} is a list of the variables in V , and the multiplication by z is represented as taking the corresponding constraint with multiplicity z . According to (8), any projection $\pi_i^{(k)} : D^k \rightarrow D$, $\pi_i^{(k)}(x_1, \dots, x_k) = x_i$ is an optimal assignment to I_f . Interpreted as tuples, we therefore have $\pi_i^{(k)} \in \text{Opt}(I)$ for $1 \leq i \leq k$. On the other hand, (8) states that f is not an optimal assignment, so $f(\pi_1^{(k)}, \dots, \pi_k^{(k)}) \notin \text{Opt}(I_f)$. In other words, $f \notin \text{Pol}(\text{Opt}(I_f))$.

For the opposite direction, assume that $f \in \text{supp}(\Gamma)$, so that (8) is unsatisfiable. Let I be an arbitrary instance of $\text{VCSP}(\Gamma)$, and let $\sigma_1, \dots, \sigma_k \in \text{Opt}(I)$ be k optimal solutions to I . Construct an instance Z of $\text{VCSP}(\Gamma)$ with variables D^k by replacing each valued constraint $\phi_i(\mathbf{x}_i)$ in I by $\phi_i(\sigma_1(\mathbf{x}_i), \dots, \sigma_k(\mathbf{x}_i))$, in Z , where $(\sigma_1(\mathbf{x}_i), \dots, \sigma_k(\mathbf{x}_i))$ is a tuple of variables in $(D^k)^{\text{ar}(\phi_i)}$. Now, if f were not an optimal solution to Z , then Z would be a solution to (8), a contradiction. Hence $f \in \text{Pol}(\text{Opt}(I))$. Since I and σ_i were chosen arbitrarily, this establishes the lemma. \square

2.3 Cores and Constants

Definition 12. Let Γ be a valued constraint language with domain D and let $S \subseteq D$. The sub-language $\Gamma[S]$ of Γ induced by S is the valued constraint language defined on domain S and containing the restriction of every weighted relation $\phi \in \Gamma$ onto S .

Definition 13. A valued constraint language Γ is a core if all unary operations in $\text{supp}(\Gamma)$ are bijections. A valued constraint language Γ' is a core of Γ if Γ' is a core and $\Gamma' = \Gamma[f(D)]$ for some $f \in \text{supp}(\omega)$ with ω a unary fractional polymorphism of Γ .

The following lemma implies that when studying the computational complexity of a valued constraint language Γ we may assume that Γ is a core.

Lemma 3. *Let Γ be a valued constraint language and Γ' a core of Γ . Then, for all instances I of $VCSP(\Gamma)$ and I' of $VCSP(\Gamma')$, where I' is obtained from I by substituting each function in Γ for its restriction in Γ' , the optimum of I and I' coincide.*

A special case of Lemma 3 for finite-valued constraint languages was proved by the authors in [27]. Lemma 3 has also been observed in [22] and in another recent paper of the authors [29].

Proof. By definition, $\Gamma' = \Gamma[f(D)]$, where D is the domain of Γ and $f \in \text{supp}(\omega)$ for some unary fractional polymorphism ω . Assume that I is satisfiable, and let σ be an optimal assignment to I . Now $f \circ \sigma$ is a satisfying assignment to I' , and by Lemma 2, $f \circ \sigma$ is also an optimal assignment to I . Conversely, any satisfying assignment to I' is a satisfying assignment to I of the same value. \square

Let $\mathcal{C}_D = \{\{(d)\} \mid d \in D\}$ be the set of constant unary relations on the set D .

Lemma 4 ([22]). *Let Γ be a core valued constraint language. The problems $VCSP(\Gamma)$ and $VCSP(\Gamma \cup \mathcal{C}_D)$ are polynomial-time equivalent.*

A special case of Lemma 4 for finite-valued constraint languages was proved by the authors in [27], building on [15], and Lemma 4 can be proved similarly; we refer the reader to [22].

3 Sherali-Adams Relaxations and Valued Relational Width

In this section, we state and prove our main result on the applicability of Sherali-Adams relaxations to VCSPs. First, we define some notions concerning *bounded relational width* which is the basis for our proof.

We write (S, C) for (valued) constraints that involve (unweighted) relations, where S is the scope and C is the constraint relation. For a tuple $\mathbf{x} \in D^S$, we denote by $\pi_{S'}(\mathbf{x})$ its projection onto $S' \subseteq S$. For a constraint (S, C) , we define $\pi_{S'}(C) = \{\pi_{S'}(\mathbf{x}) \mid \mathbf{x} \in C\}$.

Let $1 \leq k \leq \ell$ be integers. The following definition is equivalent³ to the definition of (k, ℓ) -minimality for CSP instances given in [1].

Definition 14. *A CSP-instance $J = (V, D, \{(S_i, C_i)\}_{i=1}^q)$ is said to be (k, ℓ) -minimal if:*

- *For every $S \subseteq V$, $|S| \leq \ell$, there exists $1 \leq i \leq q$ such that $S = S_i$.*
- *For every $i, j \in [q]$ such that $|S_j| \leq k$ and $S_j \subseteq S_i$, $C_j = \pi_{S_j}(C_i)$.*

There is a straightforward polynomial-time algorithm for finding an equivalent (k, ℓ) -minimal instance [1]. This leads to notion of *relational width*:

Definition 15. *A constraint language Δ has relational width (k, ℓ) if, for every instance $J \in \text{CSP}(\Delta)$, an equivalent (k, ℓ) -minimal instance is non-empty if, and only if, J has a solution.*

A k -ary idempotent operation $f : D^k \rightarrow D$ is called a *weak near-unanimity* (WNU) operation if, for all $x, y \in D$,

$$f(y, x, x, \dots, x) = f(x, y, x, x, \dots, x) = f(x, x, \dots, x, y).$$

³The two requirements in [1] are: for every $S \subseteq V$ with $|S| \leq \ell$ we have $S \subseteq S_i$ for some $1 \leq i \leq q$; and for every set $W \subseteq V$ with $|W| \leq k$ and every $1 \leq i, j \leq q$ with $W \subseteq S_i$ and $W \subseteq S_j$ we have $\pi_W(C_i) = \pi_W(C_j)$.

Definition 16 (BWC). *We say that a clone of operations satisfies the bounded width condition (BWC) if it contains WNU operations of all but finitely many arities.*

Theorem 1 ([3, 21]). *Let Δ be a constraint language of finite size containing all constant unary relations. Then, Δ has bounded relational width if, and only if, $\text{Pol}(\Delta)$ satisfies the BWC.*

Theorem 2 ([1]). *Let Δ be a constraint language. If Δ has bounded relational width, then it has relational width $(2, 3)$.*

Let $I(x_1, \dots, x_n) = \sum_{i=1}^q \phi_i(S_i)$ be an instance of the VCSP, where $S_i \subseteq V = \{x_1, \dots, x_n\}$ and $\phi_i: D^{|S_i|} \rightarrow \overline{\mathbb{Q}}$. First, we make sure that every non-empty $S \subseteq V$ with $|S| \leq \ell$ appears in some term $\phi_i(S)$, possibly by adding constant-0 weighted relations. The Sherali-Adams [24] linear programming relaxation with parameters (k, ℓ) is defined as follows. The variables are $\lambda_i(\mathbf{s})$ for every $i \in [q]$ and tuple $\mathbf{s} \in D^{S_i}$.

$$\min \sum_{i=1}^q \sum_{\mathbf{s} \in \text{Feas}(\phi_i)} \lambda_i(\mathbf{s}) \phi_i(\mathbf{s})$$

$$\lambda_j(\mathbf{t}) = \sum_{\mathbf{s} \in D^{S_i}, \pi_{S_j}(\mathbf{s}) = \mathbf{t}} \lambda_i(\mathbf{s}) \quad \forall i, j \in [q] \text{ s.t. } S_j \subseteq S_i, |S_j| \leq k, \mathbf{t} \in D^{S_j} \quad (9)$$

$$\sum_{\mathbf{s} \in D^{S_i}} \lambda_i(\mathbf{s}) = 1 \quad \forall i \in [q] \quad (10)$$

$$\lambda_i(\mathbf{s}) = 0 \quad \forall i \in [q], \mathbf{s} \notin \text{Feas}(\phi_i) \quad (11)$$

$$\lambda_i(\mathbf{s}) \geq 0 \quad \forall i \in [q], \mathbf{s} \in D^{S_i} \quad (12)$$

The $\text{SA}(k, \ell)$ optimum is always less than or equal to the VCSP optimum, hence the program is a relaxation. In anticipation of our main theorem, we make the following definition.

Definition 17. *A valued constraint language Γ has valued relational width (k, ℓ) if, for every instance I of $\text{VCSP}(\Gamma)$, if the $\text{SA}(k, \ell)$ -relaxation of I has a feasible solution, then its optimum coincides with the optimum of I .*

For a feasible solution λ of $\text{SA}(k, \ell)$, let $\text{supp}(\lambda_i) = \{\mathbf{s} \in D^{S_i} \mid \lambda_i(\mathbf{s}) > 0\}$.

Lemma 5. *Let I be an instance of $\text{VCSP}(\Gamma)$. Assume that $\text{SA}(k, \ell)$ for I is feasible. Then, there exists an optimal solution λ^* to $\text{SA}(k, \ell)$ such that, for every i , $\text{supp}(\lambda_i^*)$ is closed under every operation in $\text{supp}(\Gamma)$.*

Proof. Let ω be an arbitrary m -ary fractional polymorphism of Γ , and let λ be any feasible solution to $\text{SA}(k, \ell)$. Define λ^ω by

$$\lambda_i^\omega(\mathbf{s}) = \Pr_{f \sim \omega} [f(\mathbf{s}_1, \dots, \mathbf{s}_m) = \mathbf{s}]$$

$\mathbf{s}_1, \dots, \mathbf{s}_m \sim \lambda_i$

We show that λ^ω is a feasible solution to $\text{SA}(k, \ell)$, and that if λ is optimal, then so is λ^ω .

Clearly λ_i^ω is a probability distribution for each $i \in [q]$, so (10) and (12) hold. Since ω is a fractional polymorphism of Γ , we have $\mathbf{s} \in \text{Feas}(\phi_i)$ for any choice of $f \in \text{supp}(\omega)$ and $\mathbf{s}_1, \dots, \mathbf{s}_m \in \text{supp}(\lambda_i)$. Hence, $\lambda_i^\omega(\mathbf{s}) = 0$ for $\mathbf{s} \notin \text{Feas}(\phi_i)$, so (11) holds.

Finally, let $j \in [q]$ be such that $S_j \subseteq S_i$, $|S_j| \leq k$, and let $\mathbf{t} \in D^{S_j}$. Then,

$$\begin{aligned}
\sum_{\mathbf{s} \in D^{S_i}, \pi_{S_j}(\mathbf{s}) = \mathbf{t}} \lambda_i^\omega(\mathbf{s}) &= \sum_{\mathbf{s} \in D^{S_i}, \pi_{S_j}(\mathbf{s}) = \mathbf{t}} \Pr_{f \sim \omega} [f(\mathbf{s}_1, \dots, \mathbf{s}_m) = \mathbf{s}] \\
&= \Pr_{f \sim \omega} [\pi_{S_j}(f(\mathbf{s}_1, \dots, \mathbf{s}_m)) = \mathbf{t}] \\
&= \Pr_{f \sim \omega} [\mathbf{t}_1 = \pi_{S_j}(\mathbf{s}_1) \wedge \dots \wedge \mathbf{t}_m = \pi_{S_j}(\mathbf{s}_m) \wedge f(\mathbf{t}_1, \dots, \mathbf{t}_m) = \mathbf{t}] \\
&= \Pr_{f \sim \omega} [f(\mathbf{t}_1, \dots, \mathbf{t}_m) = \mathbf{t}] \\
&= \lambda_j^\omega(\mathbf{t}),
\end{aligned}$$

where, in the penultimate equality, we have used the fact that (9) can be read as $\lambda_j(\mathbf{t}) = \Pr_{\mathbf{s} \sim \lambda_i} [\pi_{S_j}(\mathbf{s}) = \mathbf{t}]$. It follows that (9) also holds for λ^ω , so λ^ω is feasible.

For each $i \in [q]$, we have

$$\begin{aligned}
\sum_{\mathbf{s} \in \text{Feas}(\phi_i)} \lambda_i(\mathbf{s}) \phi_i(\mathbf{s}) &= \mathbb{E}_{\mathbf{s} \sim \lambda_i} \phi_i(\mathbf{s}) = \mathbb{E}_{\mathbf{s}_1, \dots, \mathbf{s}_m \sim \lambda_i} \sum_{j=1}^m \phi_i(\mathbf{s}_j) \\
&\geq \mathbb{E}_{f \sim \omega} \phi_i(f(\mathbf{s}_1, \dots, \mathbf{s}_m)) \\
&= \sum_{\mathbf{s} \in \text{Feas}(\phi_i)} \left(\Pr_{f \sim \omega} [f(\mathbf{s}_1, \dots, \mathbf{s}_m) = \mathbf{s}] \right) \phi_i(\mathbf{s}) \\
&= \sum_{\mathbf{s} \in \text{Feas}(\phi_i)} \lambda_i^\omega(\mathbf{s}) \phi_i(\mathbf{s}).
\end{aligned}$$

Therefore, if λ is optimal, then λ^ω must also be optimal.

Now assume that λ is an optimal solution and that $\text{supp}(\lambda)$ is not closed under some operation $f \in \text{supp}(\omega)$ for $\omega \in \text{fPol}(\Gamma)$, i.e., for some $\mathbf{s}_1, \dots, \mathbf{s}_m \in \text{supp}(\lambda)$, we have $f(\mathbf{s}_1, \dots, \mathbf{s}_m) \notin \text{supp}(\lambda)$. But note that $f(\mathbf{s}_1, \dots, \mathbf{s}_m) \in \text{supp}(\lambda_i^\omega)$. Therefore, $\lambda' = \frac{1}{2}(\lambda + \lambda^\omega)$ is an optimal solution such that $\text{supp}(\lambda_i) \subsetneq \text{supp}(\lambda_i') \subseteq D^{S_i}$. For each $i \in [q]$, D^{S_i} is finite. Hence, by repeating this procedure, we obtain a sequence of optimal solutions with strictly increasing support until, after a finite number of steps, we obtain a λ^* that is closed under every operation in $\text{supp}(\Gamma)$. \square

Theorem 3. *Let Γ be a valued constraint language of finite size containing all constant unary relations. If $\text{supp}(\Gamma)$ satisfies the BWC, then Γ has valued relational width (2, 3).*

Proof. Let I be an instance of $\text{VCSP}(\Gamma)$. The dual of the $\text{SA}(k, \ell)$ relaxation can be written in the following form. The variables are z_i for $i \in [q]$ and $y_{j, \mathbf{t}, i}$ for $i, j \in [q]$ such that $S_j \subseteq S_i$, $|S_j| \leq k$, and $\mathbf{t} \in D^{S_j}$.

$$\begin{aligned}
\max \sum_{i=1}^q z_i \\
z_i \leq \phi_i(\mathbf{s}) + \sum_{j \in [q], S_j \subseteq S_i} y_{j, \pi_{S_j}(\mathbf{s}), i} - \sum_{j \in [q], S_i \subseteq S_j} y_{i, \mathbf{s}, j} \quad \forall i \in [q], |S_i| \leq k, \mathbf{s} \in \text{Feas}(\phi_i) \quad (13)
\end{aligned}$$

$$z_i \leq \phi_i(\mathbf{s}) + \sum_{\substack{j \in [q], S_j \subseteq S_i \\ |S_j| \leq k}} y_{j, \pi_{S_j}(\mathbf{s}), i} \quad \forall i \in [q], |S_i| > k, \mathbf{s} \in \text{Feas}(\phi_i) \quad (14)$$

It is clear that if I has a feasible solution, then so does the $\text{SA}(k, \ell)$ primal. Assume that the $\text{SA}(2, 3)$ -relaxation has a feasible solution. By Lemma 5, there exists an optimal primal solution λ^* such that, for every $i \in [q]$, $\text{supp}(\lambda_i^*)$ is closed under $\text{supp}(\Gamma)$. Let y^*, z^* be an optimal dual solution.

Let $\Delta = \{C_i\}_{i=1}^q \cup \{\mathcal{C}_D\}$, where $C_i = \text{supp}(\lambda_i^*)$, and consider the instance $J = (V, D, \{(S_i, C_i)\}_{i=1}^q)$ of $\text{CSP}(\Delta)$. We make the following observations:

1. By construction of λ^* , $\text{supp}(\Gamma) \subseteq \text{Pol}(\Delta)$, so Δ contains all constant unary relations and satisfies the BWC. By Theorems 1 and 2, the language Δ has relational width $(2, 3)$.
2. The constraints (9) say that if $i, j \in [q]$, $|S_j| \leq 2$ and $S_j \subseteq S_i$, then $\lambda_j^*(\mathbf{t}) > 0$ (i.e., $\mathbf{t} \in C_j$) if, and only if, $\sum_{\mathbf{s} \in D^{S_i}, \pi_{S_j}(\mathbf{s}) = \mathbf{t}} \lambda_i^*(\mathbf{s}) > 0$ (i.e., $\mathbf{t} \in \pi_{S_j}(C_i)$). In other words, J is $(2, 3)$ -minimal.

These two observations imply that J has a satisfying assignment $\sigma: V \rightarrow D$.

By complementary slackness, since $\lambda_i^*(\sigma(S_i)) > 0$ for every $i \in [q]$, we must have equality in the corresponding rows in the dual indexed by i and $\sigma(S_i)$. We sum these rows over i :

$$\sum_{i=1}^q z_i^* = \sum_{i=1}^q \phi_i(\sigma(S_i)) + \left(\sum_{i=1}^q \sum_{\substack{j \in [q], S_j \subseteq S_i \\ |S_j| \leq 2}} y_{j, \pi_{S_j}(\sigma(S_i)), i}^* - \sum_{\substack{i \in [q] \\ |S_i| \leq 2}} \sum_{j \in [q], S_i \subseteq S_j} y_{i, \sigma(S_i), j}^* \right). \quad (15)$$

By noting that $\pi_{S_j}(\sigma(S_i)) = \sigma(S_j)$, we can rewrite the expression in parenthesis on the right-hand side of (15) as:

$$\sum_{\substack{i, j \in [q], S_j \subseteq S_i \\ |S_j| \leq 2}} y_{j, \sigma(S_j), i}^* - \sum_{\substack{i, j \in [q], S_j \subseteq S_i \\ |S_i| \leq 2}} y_{j, \sigma(S_j), i}^* = 0. \quad (16)$$

Therefore,

$$\sum_{i=1}^q \sum_{\mathbf{s} \in \text{Feas}(\phi_i)} \lambda_i^*(\mathbf{s}) \phi_i(\mathbf{s}) = \sum_{i=1}^q z_i^* = \sum_{i=1}^q \phi_i(\sigma(S_i)),$$

where the first equality follows by strong LP-duality, and the second by (15) and (16).

Since I was an arbitrary instance of $\text{VCSP}(\Gamma)$, we conclude that Γ has valued relational width $(2, 3)$. \square

4 Generalisations of Known Tractable Languages

In this section, we give some applications of Theorem 3. Firstly, we show that the BWC is preserved by going to a core and the addition of constant unary relations.

Lemma 6. *Let Γ be a valued constraint language of finite size on domain D and Γ' a core of Γ on domain $D' \subseteq D$. Then, $\text{supp}(\Gamma)$ satisfies the BWC if, and only if, $\text{supp}(\Gamma' \cup \mathcal{C}_{D'})$ satisfies the BWC.*

Proof. Let μ be a unary fractional polymorphism of Γ with an operation g in its support such that $g(D) = D'$. We begin by constructing a unary fractional polymorphism μ' of Γ such that every operation in $\text{supp}(\mu')$ has an image in D' .

We will use a technique for generating fractional polymorphisms described in [17, Lemma 10]. It takes a fractional polymorphism, such as μ , a set of *collections* \mathbb{G} , which in our case will be the set of operations in the clone of $\text{supp}(\mu)$, a set of *good* collections \mathbb{G}^* , which will be operations

from \mathbb{G} with an image in D' , and an *expansion operator* Exp which assigns to every collection a probability distribution on \mathbb{G} .

The procedure starts by generating each collection $f \in \text{supp}(\mu)$ with probability $\mu(f)$, and subsequently the expansion operation Exp maps $f \in \mathbb{G}$ to the probability distribution that assigns probability $\Pr_{h \sim \mu}[h \circ f = f']$ to each operation $f' \in \mathbb{G}$. The expansion operator is required to be *non-vanishing*, which means that starting from any collection $f \in \mathbb{G}$, repeated expansion must assign non-zero probability to a good collection in \mathbb{G}^* . In our case, this is immediate, since starting from a collection f , the good collection $g \circ f$ gets probability at least $\mu(g)$ which is non-zero by assumption. By [17, Lemma 10], it now follows that Γ has a fractional polymorphism μ' with $\text{supp}(\mu') \subseteq \mathbb{G}^*$. So every operation in $\text{supp}(\mu')$ has an image in D' .

Now, we show that if $\text{supp}(\Gamma)$ contains an m -ary WNU t , then $\text{supp}(\Gamma' \cup \mathcal{C}_{D'})$ also contains an m -ary WNU. Let ω be a fractional polymorphism of Γ with t in its support. Define ω' by $\omega'(f') = \Pr_{h \sim \mu', f \sim \omega}[h \circ f = f']$. Then, ω' is a fractional polymorphism of Γ in which every operation has an image in D' , so ω' is a fractional polymorphism of Γ' . Furthermore, for any unary operation $h \in \text{supp}(\mu')$, $h \circ t$ is again a WNU, so $\text{supp}(\Gamma')$ contains an m -ary WNU t' . Next, let $h(x) = t'(x, \dots, x)$. Since Γ' is a core, the set of unary operations in $\text{supp}(\Gamma')$ contains only bijections and is closed under composition (Lemma 1). It follows that h has an inverse $h^{-1} \in \text{supp}(\Gamma')$, and since $\text{supp}(\Gamma')$ is a clone, $h^{-1} \circ t'$ is an idempotent WNU in $\text{supp}(\Gamma')$. We conclude that $h^{-1} \circ t' \in \text{supp}(\Gamma' \cup \{\mathcal{C}_{D'}\})$.

For the opposite direction, let t' be an m -ary WNU in $\text{supp}(\Gamma' \cup \{\mathcal{C}_{D'}\})$, and let ω' be a fractional polymorphism of $\Gamma' \cup \{\mathcal{C}_{D'}\}$ with t' in its support. Then, ω' is also a fractional polymorphism of Γ' . Define ω by $\omega(f) = \Pr_{h \sim \mu', f' \sim \omega'}[f'[h, \dots, h] = f]$. Then, ω is a fractional polymorphism of Γ , and, for every $h \in \text{supp}(\mu')$, the operation $t[h, \dots, h]$ is an m -ary WNU in $\text{supp}(\omega)$. We conclude that $t \in \text{supp}(\Gamma)$, which finishes the proof. \square

Hence the BWC guarantees valued relational width $(2, 3)$ also for languages not necessarily containing constant unary relations, as required by Theorem 3.

Theorem 4. *Let Γ be a valued constraint language of finite size. If $\text{supp}(\Gamma)$ satisfies the BWC, then Γ has valued relational width $(2, 3)$.*

Proof. Let D be the domain of Γ , and $D' \subseteq D$ the domain of a core Γ' of Γ . By Lemma 6 and Theorem 3, the language $\Gamma' \cup \{\mathcal{C}_{D'}\}$ has valued relational width $(2, 3)$, so clearly Γ' has valued relational width $(2, 3)$ as well. Every feasible solution to the SA(2, 3)-relaxation of an instance I' of VCSP(Γ') is also a feasible solution to the SA(2, 3)-relaxation of the corresponding instance I of VCSP(Γ). The result now follows from Lemma 3 as the optimum of I' and I coincide. \square

Secondly, we show that for any VCSP instance over a language of valued relational width $(2, 3)$ we can not only compute the value of an optimal solution but we can also find an optimal assignment in polynomial time.

Proposition 5. *Let Γ be a valued constraint language of finite size and I an instance of VCSP(Γ). If $\text{supp}(\Gamma)$ satisfies the BWC, then an optimal assignment to I can be found in polynomial time.*

Proof. Let Γ' be a core of Γ on domain D' , and let $\Gamma_c = \Gamma' \cup \{\mathcal{C}_{D'}\}$. By Lemma 6, $\text{supp}(\Gamma_c)$ satisfies the BWC, so by Theorem 3 we can obtain the optimum of I by solving a linear programming relaxation. Now, we can use self-reduction to obtain an optimal assignment. It suffices to modify the instance I to successively force each variable to take on each value of D' . Whenever the optimum of the modified instance matches that of the original instance, we can move on to assign the next variable. This means that we need to solve at most $1 + |V||D'|$ linear programming relaxations before finding an optimal assignment, where V is the set of variables of I . \square

Finally, we show that testing for the BWC is a decidable problem. We rely on the following result that was proved in [19], and also follows from results in [1].

Theorem 6 ([19]). *An idempotent clone C of operations satisfies the BWC if, and only if, C contains a ternary WNU f and a 4-ary WNU g with $f(y, x, x) = g(y, x, x, x)$ for all x and y .*

Proposition 7. *Testing whether a valued constraint language of finite size satisfies the BWC is decidable.*

Proof. Let Γ be a valued constraint language of finite size on domain D . Let Γ' be a core of Γ defined on domain $D' \subseteq D$. Finding D' and Γ' can be done via linear programming [28, Section 4]. By Lemma 6, $\text{supp}(\Gamma)$ satisfies the BWC if, and only if, $\text{supp}(\Gamma' \cup \mathcal{C}_{D'})$ satisfies the BWC. As constant unary relations enforce idempotency, by Theorem 6, $\text{supp}(\Gamma' \cup \mathcal{C}_{D'})$ satisfies the BWC if, and only if, $\text{supp}(\Gamma' \cup \mathcal{C}_{D'})$ contains a ternary WNU f and a 4-ary WNU g with $f(y, x, x) = g(y, x, x, x)$ for all x and y . It is easy to write a linear program that checks for this condition, as it has been done in the context of finite-valued constraint languages [28, Section 4]. \square

4.1 Tractable Languages

Here we give some examples of previously studied valued constraint languages and show that, as a corollary of Theorem 4, they all have valued relational width $(2, 3)$.

Example 3. Let ω be a ternary fractional operation defined by $\omega(f) = \omega(g) = \omega(h) = \frac{1}{3}$ for some (not necessarily distinct) majority operations f, g , and h . Cohen et al. proved the tractability of any language improved by ω by a reduction to CSPs with a majority polymorphism [8].

Example 4. Let ω be a ternary fractional operation defined by $\omega(f) = \frac{2}{3}$ and $\omega(g) = \frac{1}{3}$, where $f : \{0, 1\}^3 \rightarrow \{0, 1\}$ is the Boolean majority operation and $g : \{0, 1\}^3 \rightarrow \{0, 1\}$ is the Boolean minority operation. Cohen et al. proved the tractability of any language improved by ω by a simple propagation algorithm [8].

Example 5. Generalising Example 4 from Boolean to arbitrary domains, let ω be a ternary fractional operation such that $\omega(f) = \frac{1}{3}$, $\omega(g) = \frac{1}{3}$, and $\omega(h) = \frac{1}{3}$ for some (not necessarily distinct) majority operations f and g , and a minority operation h ; such an ω is called an MJN. Kolmogorov and Živný proved the tractability of any language improved by ω by a 3-consistency algorithm and a reduction, via Example 6, to submodular function minimisation [18].

The following corollary of Theorem 4 generalises Examples 3-5.

Corollary 1. *Let Γ be a valued constraint language of finite size such that $\text{supp}(\Gamma)$ contains a majority operation. Then, Γ has valued relational width $(2, 3)$.*

Proof. Let f be a majority operation in $\text{supp}(\Gamma)$. Then, for every $k \geq 3$, f generates a WNU g_k of arity k : $g_k(x_1, \dots, x_k) = f(x_1, x_2, x_3)$. By Lemma 1, $\text{supp}(\Gamma)$ is a clone, so $g_k \in \text{supp}(\Gamma)$ for all $k \geq 3$. Therefore, $\text{supp}(\Gamma)$ satisfies the BWC and the result follows from Theorem 4. \square

Example 6. Let ω be a binary fractional operation defined by $\omega(f) = \omega(g) = \frac{1}{2}$, where f and g are conservative and commutative operations and $f(x, y) \neq g(x, y)$ for every x and y ; such an ω is called a *symmetric tournament pair* (STP). Cohen et al. proved the tractability of any language improved by ω by a 3-consistency algorithm and an ingenious reduction to submodular function minimisation [7]. Such languages were shown to be the only tractable languages among conservative finite-valued constraint languages [18].

The following corollary of Theorem 4 generalises Example 6.

Corollary 2. *Let Γ be a valued constraint language of finite size such that $\text{supp}(\Gamma)$ contains two symmetric tournament operations (that is, binary operations f and g that are both conservative and commutative and $f(x, y) \neq g(x, y)$ for every x and y). Then, Γ has valued relational width $(2, 3)$.*

Proof. It is straightforward to verify that $h(x, y, z) = f(f(g(x, y), g(x, z)), g(y, z))$ is a majority operation, as observed in [7, Corollary 5.8]. The claim then follows from Corollary 1. \square

Example 7. Generalising Example 6, let ω be a binary fractional operation defined by $\omega(f) = \omega(g) = \frac{1}{2}$, where f and g are conservative and commutative operations; such an ω is called a *tournament pair*. Cohen et al. proved the tractability of any language improved by ω by a consistency-reduction relying on Bulatov's result [4], which in turn relies on 3-consistency, to the STP case from Example 6 [7].

The following corollary of Theorem 4 generalises Example 7.

Corollary 3. *Let Γ be a valued constraint language of finite size such that $\text{supp}(\Gamma)$ contains a tournament operation (that is, a binary conservative and commutative operation). Then, Γ has valued relational width $(2, 3)$.*

Proof. Let f be a tournament operation from $\text{supp}(\Gamma)$. We claim that f is a 2-semilattice; that is, f is idempotent, commutative, and satisfies the restricted associativity law $f(x, f(x, y)) = f(f(x, x), y)$. To see that, notice that $f(x, f(x, y)) = x$ if $f(x, y) = x$ and $f(x, f(x, y)) = y$ if $f(x, y) = y$; together, $f(x, f(x, y)) = f(x, y)$. On the other hand, trivially $f(f(x, x), y) = f(x, y)$. Also note that $f(x, f(y, x)) = f(x, f(x, y)) = f(x, y)$, so f is a ternary WNU. For every $k \geq 3$, f generates a WNU g_k of arity k : $g_k(x_1, \dots, x_k) = f(f(\dots(f(x_1, x_2), x_3), \dots), x_k)$. By Lemma 1, $\text{supp}(\Gamma)$ is a clone, so $g_k \in \text{supp}(\Gamma)$ for all $k \geq 3$. Therefore, $\text{supp}(\Gamma)$ satisfies the BWC so the result follows from Theorem 4. \square

Example 8. In this example we denote by $\{\{\dots\}\}$ a multiset. Let ω be a binary fractional operation on D defined by $\omega(f) = \omega(g) = \frac{1}{2}$ and let μ be a ternary fractional operation on D defined by $\mu(h_1) = \mu(h_2) = \mu(h_3) = \frac{1}{3}$. Moreover, assume that $\{\{f(x, y), g(x, y)\}\} = \{\{x, y\}\}$ for every x and y and $\{\{h_1(x, y, z), h_2(x, y, z), h_3(x, y, z)\}\} = \{\{x, y, z\}\}$ for every x, y , and z . Let Γ be a language on D such that for every two-element subset $\{a, b\} \subseteq D$, either $\omega|_{\{a, b\}}$ is an STP or $\mu|_{\{a, b\}}$ is an MJN. Kolmogorov and Živný proved the tractability of Γ by a 3-consistency algorithm and a reduction, via Example 6, to submodular function minimisation [18]. Such languages were shown to be the only tractable languages among conservative valued constraint languages [18]. We will discuss conservative valued constraint languages in more detail in Section 4.2.

The following corollary of Theorem 4 covers Example 8.

Corollary 4. *Let Γ be a valued constraint language of finite size with fractional polymorphisms ω and μ as described in Example 8. Then, Γ has valued relational width $(2, 3)$.*

Proof. Let P be the set of 2-element subsets of D such that $\omega|_{\{a, b\}}$ is an STP for $\{a, b\} \in P$ and $\mu|_{\{a, b\}}$ is an MJN for $\{a, b\} \notin P$. Let $p(x, y, z) = f(f(g(y, x), g(x, z)), g(y, z))$. Observe that $p|_{\{a, b\}}$ is a majority for $\{a, b\} \in P$, and $p|_{\{a, b\}}$ is either $\pi_1^{(3)}$ or $\pi_2^{(3)}$ for $\{a, b\} \notin P$ (possibly different projections for different 2-element subsets from P). Now let $q(x, y, z) = p(h_1(x, y, z), h_2(x, y, z), h_3(x, y, z))$. For $x, y \in \{a, b\} \in P$, $q(x, x, y) = q(x, y, x) = q(y, x, x) = p(\{\{x, x, y\}\}) = x$. For $x, y \in \{a, b\} \notin P$, $q(x, x, y) = q(x, y, x) = q(y, x, x) = p(x, x, y) = x$ as p is either the first or the second projection. Thus, q is a majority operation. The claim then follows from Corollary 1. \square

4.2 Dichotomy for Conservative Valued Constraint Languages

A valued constraint language Γ is called *conservative* if Γ contains all unary $\{0, 1\}$ -valued weighted relations. Kolmogorov and Živný gave a dichotomy theorem for such languages, showing that they are either NP-hard, or tractable, cf. Example 8. Here we prove this dichotomy using the SA(2, 3)-relaxation as the algorithmic tool.

First, we will need a technical lemma showing that the Opt operator preserves tractability.

Lemma 7. *Let Γ be a valued constraint language and I an instance of $VCSP(\Gamma)$. Then, $VCSP(\Gamma \cup \{\text{Opt}(I)\})$ polynomial-time reduces to $VCSP(\Gamma)$.*

Proof. Let $\Gamma' = \Gamma \cup \{\text{Opt}(I)\}$. Let $J' = \sum_{i=1}^q \phi_i(\mathbf{x}_i)$ be an arbitrary instance of $VCSP(\Gamma')$. We will create an instance J of $VCSP(\Gamma)$ such that if the optimum of J is too large, then J' is not satisfiable, and otherwise the optimum of J' can be computed from the optimum of J . The variables of J are the same as for J' . Let $\phi_i(\mathbf{x}_i)$ be any valued constraint in J' . If $\phi_i \in \Gamma$ then we add the valued constraint $\phi_i(\mathbf{x}_i)$ to J . Otherwise $\phi_i = \text{Opt}(I)$. In this case, we add C copies of $I(\mathbf{x}_i)$ to the instance J , where C is a number that will be chosen large enough so that if J' is satisfiable, then in any optimal assignment to J , the variables \mathbf{x}_i will be forced to be an optimal solution to the instance I . In such a solution, $\mathbf{x}_i \in \text{Opt}(I)$, and we can recover an optimal solution to J' .

The value of C is chosen as follows: if I does not have any sub-optimal satisfying assignment, then let $C = 1$. Otherwise, let $C = \lceil (U - L + 1)/\Delta \rceil$, where U is an upper bound on the optimal value of J' , L is a lower bound on the optimal value of J' , and Δ is the least difference between a sub-optimal and an optimal assignment to I . Both U and L can be computed in polynomial time by taking the sum of the largest, respectively smallest, finite values of each valued constraint. The value of C depends linearly on the number of constraints in J' , so the size of J is polynomial in the size of J' .

Let $\min(J)$, $\min(J')$, and $\min(I)$ denote the optimal value of the respective instance. Assume first that J' has a satisfying assignment. Then, this assignment is also a satisfying assignment to J , so

$$\min(J) \leq CN \min(I) + \min(J'), \quad (17)$$

where N is the number of occurrences of $\text{Opt}(I)$ in J' .

If J has a satisfying assignment σ , then we distinguish two cases. First, assume that σ assigns an optimal value to every copy of I . Then, σ is also a satisfying assignment of J' , so

$$\min(J') \leq \text{Val}(\sigma) - CN \min(I), \quad (18)$$

where $\text{Val}(\sigma)$ denotes the value of σ . From (17) and (18), we see that if σ is an optimal assignment to J , so that $\text{Val}(\sigma) = \min(J)$, then it is also an optimal assignment to J' .

Otherwise, σ assigns a sub-optimal value to at least C copies of I , so

$$\text{Val}(\sigma) \geq C(\min(I) + \Delta) + C(N - 1) \min(I) + \min(J') \geq (U - L + 1) + CN \min(I) + L. \quad (19)$$

In this case, $\min(J) > CN \min(I) + U \geq CN \min(I) + \min(J')$, so by (17), we see that J' cannot be satisfiable.

In summary, if J is unsatisfiable, or if $\min(J) > CN \min(I) + U$, then J' is unsatisfiable, and otherwise $\min(J') = \min(J) - CN \min(I)$. \square

The following theorem was proved by Takhanov [25] with a reduction, essentially amounting to Lemma 7, added in [18].

Theorem 8 ([18, 25]). *Let Γ be a conservative valued constraint language. If $\text{Pol}(\Gamma)$ does not contain a majority polymorphism, then Γ is NP-hard.*

Theorem 9. *Let Γ be a conservative valued constraint language. Either Γ is NP-hard, or Γ has valued relational width $(2, 3)$.*

Proof. Let F be the set of majority operations in $\text{Pol}(\Gamma) \setminus \text{supp}(\Gamma)$. By Lemma 2, for each $f \in F$, there is an instance I_f of $\text{VCSP}(\Gamma)$ such that $f \notin \text{Pol}(\text{Opt}(I_f))$. Let $\Gamma' = \Gamma \cup \{\text{Opt}(I_f) \mid f \in F\}$. Assume that $\text{Pol}(\Gamma')$ contains a majority polymorphism f . Then, $f \notin F$, so $f \in \text{supp}(\Gamma)$. From Corollary 1, it follows that Γ has valued relational width $(2, 3)$. If $\text{Pol}(\Gamma')$ does not contain a majority polymorphism, then, since Γ is conservative, so is Γ' , and hence Γ' is NP-hard by Theorem 8. Therefore, Γ is NP-hard by Lemma 7. \square

5 Conclusions

We have shown that most previously studied tractable valued constraint languages that are not purely relational fall into the cases covered by Theorem 4. There is however one class of languages which we have not succeeded in analysing. These are the valued constraint languages improved by a so-called generalised weak tournament pair (GWTP) identified in Uppman [30]. The definition of this class is rather intricate and we pose as an open problem the question whether such languages have valued relational width $(2, 3)$.

Problem 1. Do valued constraint languages improved by a generalised weak tournament pair have valued relational width $(2, 3)$?

References

- [1] Libor Barto. The collapse of the bounded width hierarchy. *Journal of Logic and Computation*, 2014.
- [2] Libor Barto and Marcin Kozik. Robust Satisfiability of Constraint Satisfaction Problems. In *Proceedings of the 44th Annual ACM Symposium on Theory of Computing (STOC'12)*, pages 931–940. ACM, 2012.
- [3] Libor Barto and Marcin Kozik. Constraint Satisfaction Problems Solvable by Local Consistency Methods. *Journal of the ACM*, 61(1), 2014. Article No. 3.
- [4] Andrei Bulatov. Combinatorial problems raised from 2-semilattices. *Journal of Algebra*, 298:321–339, 2006.
- [5] Andrei Bulatov, Andrei Krokhin, and Peter Jeavons. Classifying the Complexity of Constraints using Finite Algebras. *SIAM Journal on Computing*, 34(3):720–742, 2005.
- [6] Chandra Chekuri, Sanjeev Khanna, Joseph Naor, and Leonid Zosin. A linear programming formulation and approximation algorithms for the metric labeling problem. *SIAM Journal on Discrete Mathematics*, 18(3):608–625, 2004.
- [7] David A. Cohen, Martin C. Cooper, and Peter G. Jeavons. Generalising submodularity and Horn clauses: Tractable optimization problems defined by tournament pair multimorphisms. *Theoretical Computer Science*, 401(1-3):36–51, 2008.

- [8] David A. Cohen, Martin C. Cooper, Peter G. Jeavons, and Andrei A. Krokhin. The Complexity of Soft Constraint Satisfaction. *Artificial Intelligence*, 170(11):983–1016, 2006.
- [9] Víctor Dalmau, Andrei Krokhin, and Rajsekar Manokaran. Towards a characterization of constant-factor approximable Min CSPs. In *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA’15)*. SIAM, 2015.
- [10] Víctor Dalmau and Andrei A. Krokhin. Robust Satisfiability for CSPs: Hardness and Algorithmic Results. *ACM Transactions on Computation Theory*, 5(4), 2013. Article No. 15.
- [11] Wenceslas Fernandez de la Vega and Claire Kenyon-Mathieu. Linear programming relaxations of maxcut. In *Proceedings of the 18th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA’07)*, pages 53–61. SIAM, 2007.
- [12] Alina Ene, Jan Vondrák, and Yi Wu. Local distribution and the symmetry gap: Approximability of multiway partitioning problems. In *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA’13)*, pages 306–325. SIAM, 2013.
- [13] Tomás Feder and Moshe Y. Vardi. The Computational Structure of Monotone Monadic SNP and Constraint Satisfaction: A Study through Datalog and Group Theory. *SIAM Journal on Computing*, 28(1):57–104, 1998.
- [14] Peter Fulla and Stanislav Živný. A Galois Connection for Valued Constraint Languages of Infinite Size. In preparation, 2015.
- [15] Anna Huber, Andrei Krokhin, and Robert Powell. Skew bisubmodularity and valued CSPs. *SIAM Journal on Computing*, 43(3):1064–1084, 2014.
- [16] Peter Jeavons, Andrei Krokhin, and Stanislav Živný. The complexity of valued constraint satisfaction. *Bulletin of the European Association for Theoretical Computer Science (EATCS)*, 113:21–55, 2014.
- [17] Vladimir Kolmogorov, Johan Thapper, and Stanislav Živný. The power of linear programming for general-valued CSPs. *SIAM Journal on Computing*, 2015. To appear, arXiv:1311.4219v3.
- [18] Vladimir Kolmogorov and Stanislav Živný. The complexity of conservative valued CSPs. *Journal of the ACM*, 60(2), 2013. Article No. 10.
- [19] Marcin Kozik, Andrei Krokhin, Matt Valeriote, and Ross Willard. Characterizations of several Maltsev Conditions. *Algebra Universalis*, 2014. To appear.
- [20] Gábor Kun, Ryan O’Donnell, Suguru Tamaki, Yuichi Yoshida, and Yuan Zhou. Linear programming, width-1 CSPs, and robust satisfaction. In *Proceedings of the 3rd Innovations in Theoretical Computer Science (ITCS’12)*, pages 484–495. ACM, 2012.
- [21] Benoit Larose and László Zádori. Bounded width problems and algebras. *Algebra Universalis*, 56:439–466, 2007.
- [22] Joanna Ochremiak. Algebraic properties of valued constraint satisfaction problem. Technical report, March 2014. arXiv:1403.0476.
- [23] Prasad Raghavendra. Optimal algorithms and inapproximability results for every CSP? In *Proceedings of the 40th Annual ACM Symposium on Theory of Computing (STOC’08)*, pages 245–254. ACM, 2008.

- [24] H. D. Sherali and W. P. Adams. A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems. *SIAM Journal of Discrete Mathematics*, 3(3):411–430, 1990.
- [25] Rustem Takhanov. A Dichotomy Theorem for the General Minimum Cost Homomorphism Problem. In *Proceedings of the 27th International Symposium on Theoretical Aspects of Computer Science (STACS'10)*, pages 657–668, 2010.
- [26] Johan Thapper and Stanislav Živný. The power of linear programming for valued CSPs. In *Proceedings of the 53rd Annual IEEE Symposium on Foundations of Computer Science (FOCS'12)*, pages 669–678. IEEE, 2012.
- [27] Johan Thapper and Stanislav Živný. The complexity of finite-valued CSPs. In *Proceedings of the 45th ACM Symposium on the Theory of Computing (STOC'13)*, pages 695–704. ACM, 2013.
- [28] Johan Thapper and Stanislav Živný. The complexity of finite-valued CSPs. Technical report, February 2015. arXiv:1210.2977v3. An extended abstract appeared in Proc. STOC'13, submitted for publication.
- [29] Johan Thapper and Stanislav Živný. Necessary Conditions on Tractability of Valued Constraint Languages. Technical report, February 2015. Submitted for publication, arXiv:1502.03482.
- [30] Hannes Uppman. The Complexity of Three-Element Min-Sol and Conservative Min-Cost-Hom. In *Proceedings of the 40th International Colloquium on Automata, Languages, and Programming (ICALP'13)*, volume 7965 of *Lecture Notes in Computer Science*, pages 804–815. Springer, 2013.
- [31] Yuichi Yoshida and Yuan Zhou. Approximation schemes via Sherali-Adams hierarchy for dense constraint satisfaction problems and assignment problems. In *Innovations in Theoretical Computer Science (ITCS'14)*, pages 423–438. ACM, 2014.